① $(x+\delta x)\cdot x^{-1} = v^\alpha_\beta(x)\,\delta x^\beta$ $\qquad v^\alpha_\beta(x) = \left.\dfrac{\partial \psi^\alpha}{\partial x^\beta}\right|_{y=\varphi(x)}$

$(\psi+\delta\psi)\,\psi^{-1} = v^\alpha_\beta(\psi)\,\delta\psi^\beta$

$$\underbrace{(x+\delta x)\cdot y}_{= \psi+\delta\psi}\cdot(x\cdot y)^{-1} = (x+\delta x)\cdot x^{-1} = v^\beta_\alpha(x)\,\delta x^\beta$$

$$\Rightarrow \quad V^\alpha_\gamma(\psi)\,\boxed{\delta\psi^\gamma = \frac{\partial \psi^\gamma}{\partial x^\beta}\,\delta x^\beta} = V^\alpha_\beta(x)\,\delta x^\beta$$

$$\Rightarrow \quad \boxed{V^\alpha_\gamma(\psi)\cdot\frac{\partial\psi^\gamma}{\partial x^\beta} = V^\alpha_\beta(x)} \iff \boxed{\frac{\partial\psi^\gamma}{\partial x^\beta} = u^\gamma_\alpha(\psi)\,v^\alpha_\beta(x)}$$

This is how to derive (PDE-1) in the last lecture via the infinitesimal notations.

② Lie Subgroups & subalgebra.

Def 2a. $H \subset G$ is a Lie subgroup if the inclusion

(1) It is a subgroup & $i: H \to G$ is a smooth immersion.

2b. $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra if it is a linear subspace & closed under $[\ ,\ ]$

$$\begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i + \dots$$

Remark: (i) $[x, y]$ is NOT associative

e.g. $[[j, i], i] = [-k, i] = -j$

$[j, [i, i]] = 0$

i, j, k in the $\times$ of $\mathbb{R}^3$ (Cross product)

$i = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $j = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $k = \begin{pmatrix}0\\0\\1\end{pmatrix}$

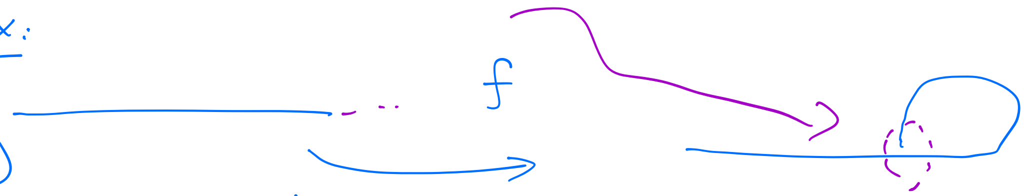
(ii) We say $f: M \to N$ is a regular submanifold if

$f: M \to N$

(i) df is injective, & (ii) f is 1-1, & $f: M \to f(M) \subset N$ (iii) is a homeomorphism with induced topo.

Ex:

(i)



is NOT a regular embedding.

(ii) (t, at) a irrational in T^2

is NOT regular embedding.

The reason of defining Lie subgroup in the above manner is due to Frobenius Theorem:

Let $\mathcal{D} = $ locally $\text{Span}\{\underline{X_1(x), \dots, X_k(x)}\}$, $\{X_i\}$ tangent vectors linearly $\forall x \in M$ independent & smooth, a k-dim distribution on M such that it is closed w.r.t $[\ ,\ ]$, namely $\forall X, Y \in \mathcal{D}$.

$[X, Y] \in \mathcal{D}$. integrable

Then $\exists !$ a maximum integral sub manifold N (of dimension k) passing any given point $p \in M$ & $\forall x \in N$ $T_xN = \mathcal{D}(x)$.

This integral manifold may NOT be embedded! ["quasi-regularity" of Frobenius Thm by some authors]

Theorem 1: $\exists$ 1-1 correspondence between $\underset{\text{Lie subalgebra}}{\eta \subset \mathfrak{g}}$ [Given G & $\mathfrak{g}$]

& H - connected Lie subgroups.

Pf: $\text{Lie}(H)$ denote the Lie algebra of H.

$\imath : H \hookrightarrow G$

$(\imath)_* \ \text{Lie}(H) \hookrightarrow \text{Lie}(G)$ is a Lie algebra homomorphism.

In general. $\varphi : H \to G$ a Lie group homomorphism $\Rightarrow \varphi_* : \eta \to \mathcal{G}$ is a Lie algebra homomorphism

$\varphi_* : \eta \to \mathfrak{g}$

$[\varphi_*(x), \varphi_*(y)] = \varphi_*([x,y])$

Smooth &

namely $\varphi_*([x_1, x_2]) = [\varphi_*(x_1), \varphi_*(x_2)]$

Pf: $\boxed{\varphi(g \cdot \sigma) = \varphi(g) \cdot \varphi(\sigma)}$

$\Rightarrow L_{\varphi(g)} \circ \varphi^{(\sigma)} = \varphi \cdot L_g(\sigma) = \varphi(g\sigma)$

$\text{''} \ \varphi(g)\varphi(\sigma)$

Hence if $X \in \eta$ is left invariant $\Rightarrow \varphi_*(X(e)) = Y(e)$

& Y is a left invariant vector field of G

$(L_{\varphi(g)})_* \cdot \varphi_* = (\varphi)_* \cdot (L_g)_*$

will have

$$\boxed{Y(\varphi(g))} = (L_{\varphi(g)})_*(Y(e)) = (L_{\varphi(g)})_* \varphi_*(X(e))$$

$$= \varphi_* \circ (L_g)_*(X(e)) = \boxed{\varphi_*(X(g))}$$

$\Rightarrow X, \& Y$ as above are φ - related

Using $\varphi_*([X_1, X_2]) = [\varphi_*(X_1), \varphi_*(X_2)]$ (☆)

$$\Rightarrow \varphi_*(\underbrace{[x_1, x_2]}^{\in T_eH}) = \varphi_*([X_1, X_2]_e)$$

$$\overset{(☆)}{=} [\varphi_*(X_1), \varphi_*(X_2)]_e$$

$$= [Y_1, Y_2]_e = [\varphi_*(x_1), \varphi_*(x_2)].$$

Conversely, given $\eta \subset \mathfrak{g}$, a subalgebra,

which we may view as a subspace of T_eG. &

$\mathrm{span}\{x_1, \dots x_k\}$ $\{x_i\}$ linearly independent

it extends to G by the left-translation

$$X_i(g) = (L_g)_*(x_i)$$

Namely η is the space $\mathrm{span}\{X_1 \cdots X_k\}$

Moreover η is a subalgebra means

$$[X_i, X_j] \in \eta.$$

Now $\mathcal{D}|_g = \mathrm{span}\{X_1(g), \dots X_k(g)\}$

is a distribution, smoothly depends on g

$$\forall \quad X = \sum_{i=1}^{k} a_i(g) X_i(g), \quad Y = \sum b_i(g) X_i(g)$$

$$\Rightarrow [X, Y] = \sum a_i(g) b_j(g) [X_i, X_j]$$

$$+ \sum a_i [X_i(b_j)] X_j - \sum b_i X_i(a_j) X_j$$

$$\Rightarrow [X, Y] \in \mathcal{D}. \quad \text{Since } [X_i, X_j]_{(g)} \in \mathcal{D}(g).$$

$\forall g.$ due to the assumption η is a sub-algebra

$\Rightarrow \mathcal{D}$ is integrable

Let H be the maximum integration mfld passing $e \in G$

Since $L_g H$ will be the $\underset{\text{maximum}}{\wedge}$ integration mfld passing g

Since $\mathcal{D}$ is L_g invariant.

$L_g : G \to G$
smooth
invertible

$\Rightarrow \forall h \in H$ $\quad h^{-1}H$ - passing e

" special $g = h^{-1}$

& H - passing e

$\Rightarrow \quad H = h^{-1}H \quad \forall h \in H$

$\Rightarrow \quad H = \bigcup_{h \in H} h^{-1}H \quad \Rightarrow \quad \underbrace{H^{-1}H = H}$

We conclude that H is closed under $\mathfrak{g}$-inversion

& - multiplication

$$h_1 \cdot h_2 = h_1 \cdot (h_2^{-1})^{-1} = h_1 (h_2')^{-1} \in H$$

 " h_2'

Namely, the $\underset{\text{maximum}}{\wedge}$ integration submanifold H is a group.

(3) Lie Groups $\longleftrightarrow$ Lie algebras

(local, sub Lie groups) connected

Establish the identification as two categories

The next nature question is: Can the "morphisms" be identified in a 1-1-manner

Answer is "No". strictly speaking. But Almost true.

Ex 1.22. 5):
$\quad G_1 = S^1, \quad G_2 = \mathbb{R}$

$\mathfrak{g}_1 = \mathfrak{g}_2 = \mathbb{R}; \quad \mu_c \quad 1 \longmapsto c$

But $\phi : G_1 \to G_2$ must be zero. map

$\Rightarrow \quad \nexists \ \phi \quad (\phi_*) = \mu_c.$

$$\boxed{\begin{array}{l}\underline{\text{Proposition}}: \text{ If } G_1 \text{ is connected, } \quad \phi : G_1 \longrightarrow G_2 \\ \quad \text{is decided by } \quad (\phi)_* : \mathfrak{g}_1 \to \mathfrak{g}_2\end{array}}$$

W.L.G we may also assume G_2 is connected since

$$\phi(G_1) \subset (G_2)_0 \text{ — the component containing } e.$$

$$\mathrm{Graph}(\phi) \subset \boxed{G_1 \times G_2} \text{ is a Lie subgroup.}$$

Since $\quad (g_1, \phi(g_1)) \cdot (g_2 \quad \phi(g_2))$

$$= (g_1 g_2 \,,\, \phi(g_1) \cdot \phi(g_2)) = (g_1 g_2 \,,\, \phi(g_1 g_2)) \in \mathrm{Graph}(\phi)$$

& it is a smooth submanifold.

$$\boxed{\mathrm{Lie}(\mathrm{Graph}(\phi))} = \boxed{\mathrm{Graph}(\phi_*)} \subset \mathfrak{g}_1 \times \mathfrak{g}_2$$

Since

$G(t) = (\exp(tX), \quad \phi(\exp(tX))) = (\exp(tX), \ \exp(t\,\phi_*(X)))$

is the all possible 1-parameter subgroups passing (e, e)

$$\left.\frac{d\,G(t)}{dt}\right|_{t=0} = (X, \quad \phi_*(X)) \in \mathfrak{g}_1 \times \mathfrak{g}_2$$

Since $\mathrm{Lie}(\mathrm{Graph}(\phi))$ decides $\mathrm{Graph}(\phi)$

$\Rightarrow \quad \phi_*$ uniquely decides ϕ.

Namely. $\underset{G_1 \to G_2}{\phi} \longrightarrow \underset{\mathfrak{g}_1 \to \mathfrak{g}_2}{\phi_*}$ is a 1-1 map. if G_1 is connected

Applications: $\forall h \in G$ $\quad a_h : G \to G$ is defined by $g \to hgh^{-1}$ (conjugate action)

$da_h = (a_h)_* : \mathfrak{g} \to \mathfrak{g}$ is a linear map which is also a Lie algebra homomorphism

namely, $Ad_h : \mathfrak{g} \to \mathfrak{g}$

This provides 1st approximation of the group structure.

$$Ad : \begin{array}{ccc} G & \longrightarrow & \boxed{GL(n, \mathfrak{g})} \\ \rotatebox{90}{$\in$} & & \\ h & \longrightarrow & Ad_h \end{array}$$

Adjoint Representation of G

Apply the above, Ad is decided by $(Ad)_* = ad$.

$$ad : \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{gl}(n, \mathfrak{g}) \\ \boxed{X} & \longrightarrow & \boxed{[X, Y]} \end{array}$$

is a Lie algebra $\overset{[A,B]=AB-BA}{\text{homomorphism}}$

$$ad_{[X,Y]} = \underbrace{[ad_X, ad_Y]}_{\in \mathfrak{gl}(n, \mathfrak{g})}$$

$\mathfrak{g} \to \mathfrak{g}$

Hence Study of Lie Group (connected) can be reduced to

2nd approximation.

Study the map: $ad : \mathfrak{g} \to \mathfrak{gl}(n, \mathfrak{g})$

This is the adjoint representation of the Lie algebra.

(4) Existence of global Lie groups.

Lecture Week 1 shows that given a Lie algebra

$\exists$ a local Lie Group G, such that $\mathfrak{g} \simeq$ the given one

Globally, A theorem of Ado: $\exists$ $\forall \mathfrak{g}$ Lie algebra. $\mathfrak{g}$ is a sub-Lie algebra of $\mathfrak{gl}(N, V)$

$\Rightarrow \exists\ G \subset GL(N, V)$ such that

$\mathrm{Lie}(G) = \mathfrak{g}$

$\mathrm{Lie}(GL(N, U))$

$N = \dim(U)$

all linear transformations of $U \to V$

Namely: $\forall \rho: \mathfrak{g}_1 \to \mathfrak{g}_2$

Thm 2: $\exists\ \phi: G_1 \to G_2$ such that $\phi_* = \rho$ if $\pi_1(G_1) = \{0\}$.

5 To have the existence of ϕ given $\phi_*: \mathfrak{g}_1 \to \mathfrak{g}_2$

$G_1 \to G_2$

We need to assume G_1 is simply-connected.

[Namely:

$$\begin{array}{cccc} & G_1 & \dashrightarrow & G_2 \\ & \uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\ \rho: & T_e G_1 & \longrightarrow & T_e G_2 \\ & \wr\wr & & \\ & \mathfrak{g}_1 & & \mathfrak{g}_2 \end{array}$$

For $\dashrightarrow$ become $\longrightarrow$ we need to assume $\pi_1(G_1) = \{0\}$.]

The idea is to use the graph as before.

$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$

$\mathrm{Graph}(\rho) \subset \mathfrak{g}_1 \times \mathfrak{g}_2$

linear subspace

$\parallel$

$(x, \rho(x))$

$\subset \mathfrak{g}_1 \times \mathfrak{g}_2$

$$[(x_1, \rho(x_1)), (x_2, \rho(x_2))] = ([x_1, x_2], [\rho(x_1), \rho(x_2)])$$

$$= ([x_1, x_2], \rho([x_1, x_2]))$$

$\in \mathrm{Graph}(\rho)$

Hence $\mathrm{Graph}(\rho)$ is a Lie sub-algebra.

$\Rightarrow \exists\ H \subset G_1 \times G_2$ — a subgroup such that

$\mathfrak{h} = \mathrm{Graph}(\rho)$ by Theorem 1.

Now: Let $p_1 : G_1 \times G_2 \to G_1$ is the projection

$(p_1)_* \quad \mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{g}_1$

$(p_1)_* \Big|_{\mathrm{Graph}(\rho)} \qquad (x, \rho(x)) \to x$

which is clearly an isomorphism

$\pi : X \to Y$ is called a covering map if $\forall y \in Y \ \exists U$ $\pi^{-1}(U) = \bigcup_i V_i$ disjoint union $\pi : V_i \to U$ is a homeomorphism

G_2 is assumed to be connected

Claim: $\phi : G_1 \to G_2$ if $\phi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism then ϕ is a covering map.

Now by $\pi_1(G_1) = \{0\}$ in the setting of Thm 2.

$\Rightarrow \ p_1 : H \to G_1$ is an isomorphism

$(p_1)_* \ (x, \rho(x)) \to x$

Then $p_1^{-1} : G_1 \to H$, composition gives $\phi : G_1 \to G_2$

$p_2 : H \to G_2$

Moreover $(\phi)_* (x) = \rho(x)$ by the construction!

6 Covering homomorphisms

Here we prove the claim: $\phi : G_1 \to G_2$ is a covering if $\phi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism.

Lemma 1: $\Gamma = \ker \phi$ is a normal subgroup

$\phi(g \Gamma g^{-1}) = e \Rightarrow \boxed{g \Gamma g^{-1} \subset \Gamma.} \qquad \forall g \in G$

$\shortparallel$
$\phi(g) e \phi(g)^{-1} \overset{!}{=} e$

Lemma 2 Γ is a discrete subgroup.

pf: $\exists\, U \ni e$ of G_1, $\phi: U \to \phi(U)$ is a diffeomorphism
$e \in$
$\gamma \in \Gamma$ $\gamma \neq e$
$\phi(e) = e \in G_2$

$\Rightarrow$ $\Gamma \cap U = \{e\}$.

$\exists\, V$ such that $V \cdot V^{-1} \subset U$, & $V^2 \subset U$.

$v_1, v_2 \in V$

$\Rightarrow \{\gamma V\}_{\gamma \in \Gamma}$ are disjoint, if $\exists$ $\gamma_1 v_1 = \gamma_2 v_2$
$\gamma_1 V \cap \gamma_2 V \neq \phi$,
$\Rightarrow \boxed{\gamma_2^{-1}\gamma_1 = v_2 \cdot v_1^{-1}} \in U$
$\in \Gamma$ $\quad = e$

$\Rightarrow \forall \gamma \in \Gamma$ $\quad \gamma V \cap \Gamma = \{\gamma\}$ $\quad \Rightarrow \Gamma$ is discrete.

Γ is closed. $\forall g \in G$ $g \notin \Gamma$ Clearly $\exists\, U \ni g$, $U \cap \Gamma = \phi$.

Lemma 3. A discrete normal subgroup of a connected Lie group is $\subset Z(G)$ — Center of G
$\triangleq \{c \in G \mid cg = gc, \forall g \in G\}$

pf. $g(t)$ a path from e to g $\quad g(0) = e$ $g(1) = g$

$\Rightarrow$ $g(t)\gamma g(t)^{-1} \subset \Gamma$ $\quad \forall t$.

$\Rightarrow$ $g(1)\gamma g(1)^{-1} = \gamma$ since $\quad g(0)\gamma g(0)^{-1} = \gamma$

$\Rightarrow g\gamma g^{-1} = \gamma \Rightarrow g\gamma = \gamma g$ $\quad \square$

Now. $\phi(\gamma V) = \phi(V)$ is an evenly covered neighborhood of $e \in G_2$
$\parallel \triangleq \tilde{V} \subset G_2$ $\quad e \in \tilde{V}$

Since $\phi^{-1}(\tilde{V}) = \bigcup_{\gamma \in \Gamma} \gamma V$ $\quad (*)$ $\quad \gamma V \cap \gamma' V = \phi$ if $\gamma \neq \gamma'$

$g \in$ $\quad$ for some $\gamma \in \Gamma$

$\phi(g) \in \phi(V)$ $\quad \phi(g) = \phi(v)$, for some v

$\Leftrightarrow \phi(g v^{-1}) = e \Leftrightarrow g v^{-1} = \gamma \Leftrightarrow \boxed{g = \gamma v}$

$\forall g \in G_1$ $\quad \phi(gV) = \tilde{V}_1$ is an evenly covered neighborhood of $\phi(g) \in G_2$

Since $\phi^{-1}(\tilde{V}_1) = \bigcup_{\gamma \in \Gamma} \gamma(gV)$ $\quad (*)$

$g_1 \in$

$\phi(g_1 (gv)^{-1}) = e$

$\exists\, v$
$\phi(g_1) = \phi(gv) \Leftrightarrow g_1 (gv)^{-1} = \gamma \Leftrightarrow g_1 = g v \gamma$

$\Leftrightarrow \quad g_1 = \gamma(gv)$ $\quad (\gamma \in Z(G_1))$

Moreover $\quad \phi(\gamma g V) = \phi(gV)$

& if $\quad \gamma_1 g V \cap \gamma_2 g V \neq \phi \Rightarrow$

$\gamma_1 g v_1 = \gamma_2 g v_2 \Rightarrow$ $\quad g\gamma_1 v_1 = g\gamma_2 v_2$ $\quad \Rightarrow \gamma_2^{-1}\gamma_1 = v_2 v_1^{-1}$, $\gamma_2^{-1}\gamma_1 \in \Gamma$, $v_2 v_1^{-1} \in VV^{-1} \subset U$

$\Rightarrow \gamma_1 v_1 = \gamma_2 v_2 \Rightarrow \begin{cases} v_1 = v_2 \\ \gamma_1 = \gamma_2 \end{cases}$

Lemma 4. A connected Lie group G is generated by $U (\ni e)$ $\quad VV^{-1} \subset U$, $V^2 \subset U$

Pf: $\quad \boxed{H = \bigcup_{n \geq 1} V^n}$

$V^n := \{ v_1^{\pm 1} \cdots v_n^{\pm 1} \mid v_j \in V \}$

① H is a group. $\quad h_1 h_2 \in H$ if $h_1 = v_1^{\pm 1} \cdots v_k^{\pm 1}$, $h_2 = v_{k+1}^{\pm 1} \cdots v_{k+l}^{\pm 1}$

② $\quad V \subset H$.

$\Rightarrow H$ is a manifold. $\quad gH$

H is open, $\quad H^c \quad G = H \cup (\bigcup_{gH \neq H} gH) \quad \Rightarrow H = G \quad \square$

Now ϕ is also onto, since $\tilde{V} = \phi(U)$ generates G_2 $\quad \tilde{V}$ is a neighborhood of $e \in G_2$

Namely $\forall \tilde{g} \in G_2 \quad \tilde{g} = \phi(v_1) \cdots \phi(v_n)$

$= \phi(v_1 \cdots v_n) = \phi(g)$ for some g. $\quad = v_1 \cdots v_n$

Putting together we have the claim.

⑦ $\quad G/\Gamma$.

Thm 3: Assume Γ is a discrete normal subgroup of G

$\Gamma \subset Z(G)$. Then $G \to G/\Gamma$ is a covering.

Pf (i) G/Γ is a group

(ii) Pick, U & V as above $\Longrightarrow$ $\Gamma \cap U = \{e\}$

G/Γ is a manifold near $\{e\}$.

$\rightsquigarrow \pi^{-1}(\pi(\tilde{V}))$

$\pi : G \to G/\Gamma$

Now we show that

(c) $\forall\, g \in G \quad \exists\, g \in \tilde{V}$ such that $\{\gamma \tilde{V}\}_{\gamma\in\Gamma}$ disjoint

Pf: $\boxed{\tilde{V} \doteq gV}$ $\boxed{\text{if } \gamma_1 \tilde{V} \cap \gamma_2 \tilde{V} \neq \phi \text{ with } \gamma_1 \neq \gamma_2,\ \gamma_i \in \Gamma}$ $\Rightarrow$ $\gamma_1 g v_1 = \gamma_2 g v_2$

$\Rightarrow \gamma_1 v_1 = \gamma_2 v_2 \Rightarrow \begin{cases} \gamma_1 = \gamma_2 \\ v_1 = v_2 \end{cases}$

Hence $\{\gamma \tilde{V}\}_{\gamma\in\Gamma}$ disjoint

$\Longrightarrow$ $G \to G/\Gamma$ is a topological covering.

(b) $\boxed{\forall\, g_1 \neq g_2 \ \& \ \boxed{g_1 \notin g_2\Gamma} \quad \exists\ \tilde{V}_1 \ \& \ \tilde{V}_2 \quad \gamma \tilde{V}_1 \cap \gamma' \tilde{V}_2 = \phi. \quad \forall\, \gamma, \gamma' \in \Gamma}$

Proof. Pick U such that $\Rightarrow$ Show G/Γ is Hausdorff.

$$g_2^{-1}\Gamma g_1 \cap U = \phi$$

This is possible since $g_2^{-1}\Gamma g_1$ discrete, & $e \notin \boxed{g_2^{-1}\Gamma g_1}$

Now construct V as above $V.V^{-1} \subset U$ & $V^2 \subset U$.

Claim $\boxed{g_1 V = \tilde{V}_1}$ satisfies the requirement
$\boxed{g_2 V = \tilde{V}_2}$

Since if $\gamma g_1 V \cap \gamma' g_2 V \neq \phi$

$\Rightarrow \quad \gamma g_1 v_1 = \gamma' g_2 v_2 \Rightarrow (\gamma')^{-1}\gamma g_1 v_1 = g_2 v_2 \Rightarrow$

$\Rightarrow \quad \underbrace{g_2^{-1} \gamma'' g_1}_{\in\, g_2^{-1}\Gamma g_1} = v_2 v_1^{-1} \in U$

$\Longrightarrow$ A contradiction !

(b) $\Rightarrow G/\Gamma$ is Hausdorff & it admits a manifold structure.

□ of Theorem 3

8 Cartan's theorem & Group structure on a covering

Theorem: ① $\imath : H \longrightarrow G$ is a regular embedding (w.r.t the relative topology in G) iff $\imath(H) \subset G$ is closed.

② $H \subset G$ a closed subgroup of a Lie group must be a Lie subgroup. Thm 3.21 & 3.42 of F. Werner's book.

Application: (i) $G = GL(n, \mathbb{R})$

$\Rightarrow H = \begin{cases} SO(n, \mathbb{R}) \\ O(n, \mathbb{R}) \\ SL(n, \mathbb{R}) \end{cases}$ are all Lie groups.

$SO(n,\mathbb{R})$: $\det(A) = 1$, $AA^t = A^tA = id$

$SL(n,\mathbb{R})$: $\det(A) = 1$

(ii) $G = GL(n, \mathbb{C})$ (open in $\mathbb{C}^{n^2}$), $H = \begin{cases} U(n) \\ O(n, \mathbb{C}) \end{cases}$

$U(n)$: $A\bar{A}^t = \bar{A}^tA = id$ — a real Lie group

$O(n,\mathbb{C})$: $AA^t = A^tA = id$. — a Complex Lie group

Prop: If $\pi : \tilde{G} \longrightarrow G$ is a covering (smooth), G is a Lie group. $\tilde{G}$ is a manifold, then $\tilde{G}$ admits a Lie group structure such that π is a homomorphism.

Ziller Prop 1.17 (a).

Key: product in Lie group coincides with product of $\pi_1(G)$.

$\nearrow \quad \sim$

$\Rightarrow$

$$\begin{tikzcd} & & G \arrow[d, "\pi"] \\ \tilde{G}\times\tilde{G} \arrow[r, "\pi\times\pi"] \arrow[urr, dashed] & G\times G \arrow[r] & G \end{tikzcd}$$

$$\varphi\circ(\pi\times\pi) : \tilde{G}\times\tilde{G}$$

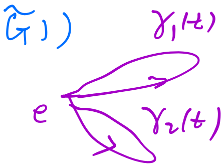
$$(\varphi)_*(\pi\times\pi)_*\left(\pi_1(\tilde{G})\times\pi_1(\tilde{G})\right) \subset \left[\pi_*(\pi_1(\tilde{G}))\right]^2$$

$$\subset \pi_*(\pi_1(\tilde{G}))$$

Hence product can be lifted.

$\gamma_1(t) \subset G$

$\gamma_2(t) \subset G$



$$\boxed{[\gamma_1]\circ[\gamma_2] = [\underbrace{\gamma_1(t)\cdot\gamma_2(t)}_{\text{Lie group product}}]}$$

The best place to read the proof of Cartan's theorem is Matsushima's Differential Manifolds. Ch 4.20.

He also did the proof of Yamabe & Freudenthal's theorem.

: Every arc-connected subgroup of a Lie group is a connected Lie group.